Statistical Mechanics Helps Us Count Alternating Sign Matrices

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Outline

Alternating Sign Matrices

Definition (Alternating Sign Matrix)

$A \in M_{n \times n}$ is an alternating sign matrix if:

- Each entry is either $-1, 0,$ or 1.
- Each row and column add up to 1.
- Nonzero entries in each row and column alternate in sign.

Robbins, Rumsey, and Mills discovered that they arise naturally from the **Dodgson condensation** algorithm to compute a determinant (1983).

Examples

Observe that this is a generalization of permutation matrices.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

Figure: The seven alternating sign matrices of size 3. Why is this a natural arrangement?

How many are there?

Problem (ASM Conjecture)

There are:

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$$

alternating sign matrices of order n.

Example: Consider order 3.

$$7 = \prod_{k=0}^{2} \frac{(3k+1)!}{(n+k)!} = \frac{1}{3!}\frac{4!}{4!}\frac{7!}{5!}$$

A first refinement

Observation: An ASM can only have one 1 in its top row, and the rest as zeros.

Question: How many ASMs of size n have a 1 in the k-th entry of top row?

Problem (Refined ASM Conjecture)

$$\frac{A_{n,k}}{A_{n,k-1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}$$

Refined ASM, visualized (Pascal's Triangle Style)

1
1 1
2 3 2
7 14 14 7
42 105 135 105 42
429 1287 2002 2002 1287 429
7436 26026 47320 56784 47320 26026 7436

Figure 1. The counts of *n*-by-*n* ASMs with a 1 at the top of column *k*.

2/2
2/3 3/2
2/4 5/5 4/2
2/5 7/9 9/7 5/2
2/6 9/14 16/16 14/9 6/2
2/7 11/20 25/30 30/25 20/11 7/2

Figure 2. The ratios of adjacent terms from Figure 1.

Proofs

"These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true" (Robbins)

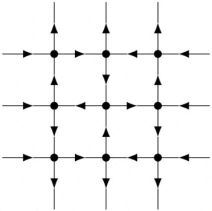
We have three proofs of the ASM conjecture.

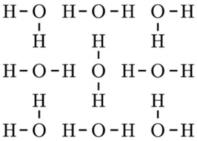
- **(Zeilberger 1992)** "Judicious assembly of existing tools": descending plane partitions, partial recurrence equations.
- **(Kuperberg 1995)**: Six-vertex model, Yang-Baxter, Izergin's determinant evaluation.
- **(Fischer 2005)** Operator Method.

The six-vertex model (”square ice”)

Statistical Mechanics: it can be useful to model particles as vertices of a lattice.

For example, the **six-vertex model**, is reminiscent of a 2-dimensional sheet of ice (hence, **square ice**).





$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Weighting

$$
\begin{array}{lccl}
\text{horizontal:} & \mathrm{H}-\mathrm{O}-\mathrm{H} & \text{has weight} & z, \\[1em]
\text{vertical:} & \begin{matrix}\mathrm{H}\\ | \\ \mathrm{O} \\ | \\ \mathrm{H}\end{matrix} & \text{has weight} & z^{-1}, \\[1em]
\begin{matrix}\text{southwest}\\ \text{northeast}\end{matrix} : & \begin{matrix}\mathrm{H}-\mathrm{O}\\ \;\;\;\;| \\ \;\;\;\;\mathrm{H}\end{matrix} \;\text{ or }\; \begin{matrix}\mathrm{H}\;\;\;\;\\ |\;\;\;\;\\ \mathrm{O}-\mathrm{H}\end{matrix} & \text{has weight} & \dfrac{az-(az)^{-1}}{a-a^{-1}}, \\[1em]
\begin{matrix}\text{southeast}\\ \text{northwest}\end{matrix} : & \begin{matrix}\mathrm{O}-\mathrm{H}\\ |\;\;\;\; \\ \mathrm{H}\;\;\;\;\end{matrix} \;\text{ or }\; \begin{matrix}\;\;\;\;\mathrm{H}\\ \;\;\;\;|\\ \mathrm{H}-\mathrm{O}\end{matrix} & \text{has weight} & \dfrac{z-z^{-1}}{a-a^{-1}}.
\end{array}
$$

For convenience, we define

$$[z] = \frac{z-z^{-1}}{a-a^{-1}}.$$

Claim: The weight of any state must be of the form $z^n[z]^{2c}[az]^{2d}$.

Define $Z_n(z, a)$ to be sum of weights of all possible states.

Kuperberg's 1995 Proof

Let $z = a = \omega = e^{2\pi i/3}$. Then $[z] = [w] = 1$ and $[az] = [\omega^2] = 1$. Any given state will evaluate to:

$$z^n[z]^{2c}[az]^{2d} = \omega^n(1)^{2n}(-1)^{2d} = \omega^n$$

So the sum over all states will be:

$$Z_n(\omega, \omega) = \omega^n A_n$$

where A_n is the number of ASMs (due to the correspondence). Apply [Izergin 1987]'s formula for Z_n.

Izergin's Formula.

We cannot apply Izergin directly. Change of variables: $z_{ij} = x_i/y_i$. Then:

$$Z_n(z, a) = Z_n(x, y, a) = \frac{\prod_{i=1}^{n} x_i/y_i \prod_{1 \le i,j \le n}[x_i/y_i][ax_i/y_j]}{\prod_{1 \le i,j \le n}[x_i/x_j][y_j/y_i]} \det M$$

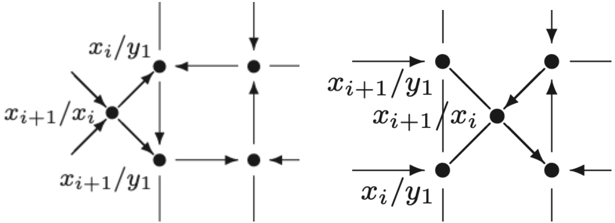
$$M_{ij} = \frac{1}{[x_i/y_j][ax_i/y_j]}$$

Kuperberg follows Izergin to prove this relation.

Proving Izergin

The crucial step is to prove that $Z_n(x, y)$ is symmetric in x_i and y_j. Prove this using **Yang-Baxter equation** (aka triangle-to-triangle).

Strategy: Attach a new vertex with label x_{i+1}/x_{i+1}. Shift it across the lattice, and reverse some of the labels. The partition function is unchanged.



General View of Yang-Baxter

Rules: $z_3 = z_1 z_2$. Need to switch labels on top and bottom labels z_2 and z_3. (This switching is where symmetry arises).

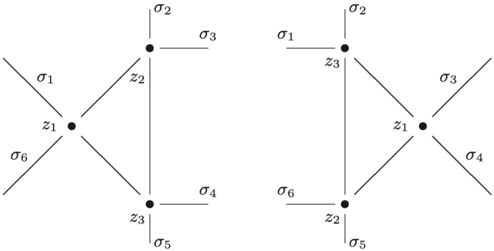


Figure 7.5. The triangle-to-triangle relation.

Zeilberger's Extension

Zeilberger revisits the problem, **uses Izergin to prove Refined ASM.**
Strategy: select weights to target the first row!

- Let $x_1 = \omega t$, the remaining $x_i = \omega$, and all $y_j = 1$. Keep $a = \omega$.

Consequences: if the 1 in the first row occurs in column r, then:

- $r - 1$ southwest molecules, with weight $[w^2 t]$.
- $n - r$ southeast molecules, with weight $[wt]$.
- Remaining southeast or northwest molecules will have weight $+1$.
- Remaining southwest or northeast molecules will have weight -1. Parity of their number same as $r - 1$.

Refined ASM Proof

$$Z_n(x', y', \omega) = \omega^n t \sum_{r=1}^{n} \left[(-1)^{r-1} [w^2 t]^{r-1} [\omega t]^{n-r} \right] A_{n,r}$$

Substitute conjectured formula for $A_{n,r}$; it works. Since the polynomials are linearly independent, if we find one solution, it is unique. $\square$

Conclusions

ASM conjecture is deceptively simple.

Six vertex model is a powerful data structure for this problem.