What is interest?

- A fundamental yet powerful concept: **interest** is the price of borrowing someone else's money. It reflects how much we think money is worth now (present value) versus some time in the future (future value).

- Can be understood via accumulation function, $a(t)$. If person A took a loan of size 1 out from person B, and we let $a(t)$ be the function representing the outstanding debt, we think of $a(t)$ as growing by default (if left untouched; of course the debt shrinks with each payment). The effective interest between two time points is described by:

  $$i_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_1)}$$

  In other words $(1 + i_{[t_1,t_2]})a(t_1) = a(t_2)$.

  Complementary to insurance is the concept of **discount**. This is measured relative to the end rather than the beginning,

  $$d_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_2)}$$

  In other words $(1 - d_{[t_1,t_2]})a(t_2) = a(t_1)$.

  In general, we can relate discount and interest as

  $$d = \frac{i}{i+1} \qquad i = \frac{d}{1-d}$$

- Simple interest is a linear relationship: $a(t) = a(0) + it$. Likewise with simple discount: $a(t) = \frac{a(0)}{1-dt}$.

  Typically we work with **compound interest**: $a(t) = (1 + i)^t a(0)$, where t is typically some integer representing fixed periods like years or months.

  Oftentimes we are given annual **nominal** interest rate compounded m times per year. Call this δ. What this means it that every $1/m$-year period, we have an accumulation factor of $(1 + \delta/m)$. So the annual **effective** interest rate is given by $(1 + \delta/m)^m - 1$.

  We can think about $m \to \infty$ as approaching a **continuous interest** regime, where we accumulate at every moment. It turns out that

  $$\lim_{m\to\infty}(1 + \delta/m)^m = e^\delta$$

If we charge interest in a continuous manner, where $e^\delta - 1$ is the annual effective interest, we call δ the **annual force of interest**.

In the continuous interest regime, it is natural to think about δ changing continuously as well.

$$i_{[0,t]} = \exp\Big(\int_0^t \delta_t dt\Big)$$

- A related notion is **inflation**. This relates to the buying power of money. In real life, inflation is calculated based on the actual price of goods in the market (like bread and smartphones), and hence is determined by the buyers and sellers of those goods. Contrast this with interest, which reflects the interaction between lenders and borrowers.

  Of course, lenders and borrowers take into account the (expected) inflation, as both parties presumably want to use their money to buy "real things". If r is the inflation rate (meaning 1 dollars now has the same purchasing power as $1 + r$ dollars in one year), then the real or **inflation-adjusted interest rate** is given by

$$i' := \frac{a(1) - (1+r)a(0)}{(1+r)a(0)} = \frac{(1+i) - (1+r)}{1+r} = \frac{i-r}{1+r}$$

  The only difference with the original definition of interest is that we compare to the inflation-adjusted version of the old value.

- Later on in these notes we will talk about the *term structure of interest rates*. This is the idea that interest rates vary depending on the term of the loan, and indeed are typically higher for longer-term loans. There are various theories for why exactly this happens (is this a "rational" behavior? or is it due to "imperfections" in the market). The intuition is more or less clear: generally lenders want their money back sooner rather than later.

Pricing

- How do you assign a price to a sequence of payments throughout time (sometimes known as a cashflow)? If $\{C_t\}$ represents a collection of payments, and we are pricing them all back to time zero using a fixed interest rate i, we can use the equation:

$$P(i) = \sum_t C_t(1+i)^{-t}$$

- Sometimes we abbreviate $(1+i)^{-1}$ as v_i and call this the **value function**.

Generic Series Equations

- For any r,
$$ar + ar^2 + ... + ar^n = \frac{a(1-r^n)}{1-r}$$
- For all $r \in [-1, 1]$,
$$ar + ar^2 + ... = \frac{a}{1-r}$$
- Arithmetic series
$$1 + 2 + ... + n = n(n+1)/2$$

Observation 1. *(arithmetico-geometric series)*

$$a(r + 2r^2 + ... + nr^n) = ar \cdot \frac{\left(\frac{r^n-1}{r-1} - nr^n\right)}{1-r}$$

Proof. Let $A = r + 2r^2 + ... + nr^n$. Let $B = 1 + r + ... + r^{n-1}$.

$$A + B = 1 + 2r + 3r^2 + ... + nr^{n-1} + nr^n$$
$$r(A + B) = A + nr^{n+1}$$
$$A(1-r) = r(B - nr^n)$$
$$A = \frac{r \cdot \left(\frac{r^n-1}{r-1} - nr^n\right)}{1-r}$$

□

- If $|r| < 1$, then as $n \to \infty$,
$$r + 2r^2 + 3r^3 + ... = \frac{r}{(1-r)^2}$$

Annuities

- An annuity is simply sequence of payments. Could be level payments, could be geometrically or arithmetically increasing. Could be repayments to a loan, or deposits into a bank account. In common parlance you hear the word annuity as it related to retirement plans: people may purchase annuities to create a stable source of income through time.

  We are often interested in pricing annuities at specific points in time. It is important to be specific about where you are in time relative to the payments.

- Consider a number line representing fixed periods of time (say month or year intervals). If the time is circled then there is a payment, say of 1 deposited into a bank account earning compound interest i with each period.

$$0 \quad \textcircled{1} \quad \textcircled{2} \quad ... \quad \textcircled{n} \quad n+1$$

  - "PV of annuity-immediate" $\implies$ cash flow is priced at time 0.
    The symbol and calculation is given by:

$$a_{\overline{n}|i} := v_i + v_i^2 + ... + v_i^n = \frac{1 - v_i^n}{i}$$

  - "PV of annuity-due" $\implies$ cash flow is priced at time 1.

$$\ddot{a}_{\overline{n}|i} := 1 + v_i + ... + v_i^{n-1} = (1+i)a_{\overline{n}|i}$$

  - "FV of annuity-immediate" $\implies$ cash flow is priced at time t.

$$s_{\overline{n}|i} := (i+1)^{n-1} + (1+i)^{n-2} + ... + 1 = (1+i)^n a_{\overline{n}|i}$$

  - "FV of annuity-due" $\implies$ cash flow is priced at time $t+1$.

$$s_{\overline{n}|i} := (i+1)^n + (1+i)^{n-1} + ... + (1+i) = (1+i)^{n+1} a_{\overline{n}|i}$$

- The PV of an arithmetically increasing annuity is given by:

$$(I_{P,Q}a)_{\overline{n}|i} = Pv + (P+Q)v^2 + ... + (P+(n-1)Q)v^n$$

Observation 2. *Arithmetic series annuities simple formulas.*

- $(I_{P,Q}a)_{\overline{n}|i} = Pa_{\overline{n}|i} + (Q/i)\Big(a_{\overline{n}|i} - nv^n\Big)$.
- $(Ia)_{\overline{n}|i} := (I_{1,1}a)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}$

Straightforward Proof. Apply Observation 1.

$$\begin{aligned}(I_{P,Q}a)_{\overline{n}|i} &= Pv(1 + v + ... + v^{n-1}) + Qv(v + ... + (n-1)v^{n-1}) \\ &= Pv \cdot \frac{1-v^n}{1-v} + Qv \cdot \frac{v\Big(\frac{v^{n-1}-1}{v-1} - (n-1)v^{n-1}\Big)}{1-v}\end{aligned}$$

Note that $\frac{v}{1-v} = \frac{1}{\frac{1}{v}-1} = \frac{1}{i}$.

$$\begin{aligned}
(I_{P,Q}a)_{\overline{n}|i} &= P \cdot \frac{1-v^n}{i} + Q \cdot \frac{\frac{v_{n-1}-1}{i} - (n-1)v^n}{i} \\
&= Pa_{\overline{n}|i} + (Q/i)(a_{\overline{n-1}|i} - (n-1)v^n) \\
&= Pa_{\overline{n}|i} + (Q/i)(a_{\overline{n-1}|i} + v^n - v^n - (n-1)v^n) \\
&= Pa_{\overline{n}|i} + (Q/i)(a_{\overline{n}|i} - nv^n)
\end{aligned}$$

If you let $P = Q = 1$, then

$$(I_{1,1}a)_{\overline{n}|i} = a_{\overline{n}|i} + a_{\overline{n}|i}/i - nv^n/i = ((i+1)/i)a_{\overline{n}|i} - nv^n/i = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}$$

□

Roundabout Proof, manipulating actuarial symbols. Recall $v = v_i = (i+1)^{-1}$ and $(Ia)_{\overline{n}|i} = v + 2v^2 + ... + (n-1)v^{n-1} + nv^n$.

$$\begin{aligned}
(Ia)_{\overline{n}|i} + \ddot{a}_{\overline{n}|i} &= 1 + 2v + 3v^2 + ... + nv^{n-1} + nv^n \\
&= (Ia)_{\overline{n}|i}v^{-1} + nv^n \\
\ddot{a}_{\overline{n}|i} - nv^n &= (Ia)_{\overline{n}|i}(v^{-1} - 1)
\end{aligned}$$

Recall that $v^{-1} - 1 = (i+1) - 1 = i$, so we obtain the desired result. As for the general increasing annuity, we have:

$$\begin{aligned}
(I_{P,Q}a)_{\overline{n}|i} &= P(v + v^2 + ... + v^n) + Q(v^2 + ... + (n-1)v^n) \\
&= Pa_{\overline{n}|i} + Qv(v + ... + (n-1)v^{n-1} + nv^n - nv^n) \\
&= Pa_{\overline{n}|i} + Qv(v + ... + (n-1)v^{n-1} + nv^n - nv^n) \\
&= Pa_{\overline{n}|i} + Qv((Ia)_{\overline{n}|i} - nv^n) \\
&= Pa_{\overline{n}|i} + Qv\left(\frac{\ddot{a}_{\overline{n}|i} - nv^n}{i} - nv^n\right) \\
&= Pa_{\overline{n}|i} + \frac{Qv}{i}\left(\ddot{a}_{\overline{n}|i} - nv^n - inv^n\right) \\
&= Pa_{\overline{n}|i} + \frac{Qv}{i}\left(\ddot{a}_{\overline{n}|i} - nv^{n-1}\right) \\
&= Pa_{\overline{n}|i} + \frac{Q}{i}\left(a_{\overline{n}|i} - nv^n\right)
\end{aligned}$$

□

(I wrote this latter proof before finding the simpler route; I suppose it was good practice with actuarial symbols...).

Observation 3. *Formula for an arithmetically decreasing annuity.*

$$(Da)_{\overline{n}|i} := v^n + 2v^{n-1} + ... + (n-1)v^2 + nv = \frac{n - a_{\overline{n}|i}}{i}$$

.

Proof. Observe that $(Da)_{\overline{n}|i} = v^{n+1}\Big((1+i) + 2(1+i)^2 + ... + n(1+i)^n\Big)$. Apply Observation 1:

$$\begin{aligned}(Da)_{\overline{n}|i} &= v^{n+1} \cdot (1+i) \cdot \frac{\frac{(1+i)^n - 1}{(1+i)-1} - n(1+i)^n}{1-(1+i)} \\ &= v^n \cdot \frac{n(1+i)^n - \frac{(1+i)^n-1}{i}}{i} = \frac{n - a_{\overline{n}|i}}{i}\end{aligned}$$

□

Loan Amortization

- An amortized loan is one where, when you make a payment, you first pay down the interest accumulated since the last payment, then you pay down the principal.
- If OB_t denotes the outstanding balance on a loan at time t; PR_t denotes the principal paid off at time t; and I_t represents the interest paid at time t, then the key formula here is:
$$I_{t+1} = i \cdot \text{OB}_t$$
(If the payments are not at fixed intervals, then the more general formula is $I_{t'} = ((1+i)^{t'-t} - 1) \cdot \text{OB}_t$ for $t' > t$).
- If you have a fixed payment schedule $K_1, ..., K_n$, where the loan is supposed to be paid back by time n, you can make some further observations:
  - The principal paid at time t is what remains of the payment after paying the interest:
  $$\text{PR}_t = K_t - I_t$$

  - The outstanding balance after payment is the previous outstanding balance, minus the principal payment:

$$\mathrm{OB}_{t+1} = \mathrm{OB}_t - \mathrm{PR}_{t-1}$$

    This comes from the fact that $\mathrm{OB}_{t+1} = (1+i)\,\mathrm{OB}_t - K_t$.

- For general payment schedules, if you want to fill out $\{I_t, \mathrm{PR}_t\}$ for all t, you need to follow the recursion induced by the previous three equations, with boundary conditions: $\mathrm{OB}_0 = L$ (the starting amount of the loan) and $\mathrm{OB}_n = 0$ (after n periods the loan is paid off).

- For fixed payment schedules, you have more direct formulas. Of course you can relate the payment value to the loan amount L through an annuity (assume we have an annuity-immediate, and n periods).

$$K_t = L/a_{\overline{n}|i} := K$$

Observation 4 (level payment amortization)**.** *Suppose I have a loan L which I take out now, which I pay off with an amortized level payment over n periods (end of period payments) with interest rate i. Then there is a simple form for the interest and principal paid off at each time.*

$$\mathrm{PR}_t = K v_i^{n-t+1} \qquad I_t = K(1 - v_i^{n-t+1})$$

Note that the principal paid off increases exponentially: $\mathrm{PR}_{t+1} = (1+i)\,\mathrm{PR}_t$.

Proof. Note that $\mathrm{OB}_t = K a_{\overline{n-t}|i}$, as there are $n-t$ payments remaining. Then note

$$I_t = i\mathrm{OB}_{t-1} = iKa_{\overline{n-(t-1)}|i} = iK\frac{1 - v_i^{n-t+1}}{i} = K(1 - v_i^{n-t+1})$$

For the principal, recall $K = \mathrm{PR}_t + I_t$. □

Bonds

- Bonds are essentially loans issued by an entity (like a government or company) to raise short-term funds. When you buy a bond, you collect *coupon payments* and then a *redemption payment* when the bond matures. To visualize this:

  - At time 0, you buy the bond.
  - At times that are circled, $\{1, ..., n\}$, you receive a coupon payment.
  - At the final time, denoting *maturity* of the bond, we receive a final coupon payment and the redemption value.

- There are many concepts and numbers associated with bonds: F is the face value. C is the redemption value (how much the buyer get at maturity). r is the coupon rate per payment period, so Fr is how much money you actually get per period. m is the number of periods.

  When a bond is **par value** or **redeemable at par**, this means $F = C$.

- There are three "perspectives" on bonds, so to speak, captured by the following equation:

$$Fr = Cg = Gj$$

  - (coupon rate) r is the coupon rate.
  - (adjusted coupon rate) g is the **adjusted coupon rate**, $g = (F/C)r$. If the bond is at par, $g = r$. We define this because it is useful.
  - (yield rate) j is the yield rate; it captures the growth rate of money put into the bond (i.e. every dollar invested in the bond appreciated by a factor $1 + j$ per period). G, referred to as the **base value**, is then defined as the PV of the coupon payments viewed as a perpetuity: Fr/j.

- There are a few ways to price bonds, i.e. evaluate their present value. The most basic equation is

$$P = Fra_{\overline{n}|j} + Cv_j^n \tag{1}$$

Observation 5. *Other pricing formulas for bonds:*

- *(price-discount)* $P = C(g - j)a_{\overline{n}|j} + C$
- *(base value)* $P = (C - G)v_j^n + G$.
- *(Makeham's formula)* $P = (g/j)(C - Cv_j^n) + Cv_j^n$.

Proof. For price-discount, replace Fr with Cg.

$$P = Cga_{\overline{n}|j} + Cv_j^n = Cga_{\overline{n}|j} + \frac{Cjv_j^n + Cj - Cj}{j} = Cga_{\overline{n}|j} - Cja_{\overline{n}|j} + C$$

For base value, replace Fr with Gj.

$$P = Gja_{\overline{n}|j} + Cv_j^n = G(1 - v_j^n) + Cv_j^n$$

For Makeham's, use $Fr = Cg$ and just distribute:

$$P = Cga_{\overline{n}|j} + Cv_j^n = C(g/j)(1 - v_j^n) + Cv_j^n = (g/j)(C - Cv_j^n) + Cv_j^n$$

□

The nice thing about Makeham's formula is that if you know the present value of the redemption, you can price the bond without knowing how many terms it lasts.

Callable Bonds

- A bond is callable if the issuer has some choice of dates when they can repay the debt. This of course puts more power in the issuer's hands, and makes the bond less desirable to the purchaser.
- A bond has a different yields based on when it is called in. A rational investor should assume the worst-case yield. In other words, you should assume the issuer will call on the date that induces the *lowest yield*.

Definition 1. *The yield rate of a callable bond is the lowest yield rate induced by the various potential call dates.*

Observation 6. *If a bond is bought at a discount, then the yield rate of that bond is the yield induced by the earliest call date. More generally, if $t_1 < t_2 < \ldots$ and y_i is the yield induced by calling at t_i,*

- *(discount)* $j_1 > j_2 > \ldots > \boxed{j_n}$
- *(premium)* $\boxed{j_1} < j_2 < \ldots < j_n$

Proof. Recall the basic bond pricing formula. Let j_k be the yield rate corresponding to calling at period j. Let $P_k(j_k)$ denote the (fair) price of the bond if called at period k with yield j_k. Observe:

$$\begin{aligned}
P_k(j_k) &= (Cg)a_{\overline{k}|j_k} + Cv_{j_k}^k \\
&= (Cg)(a_{\overline{k-1}|j_k} + v_{j_k}^k) + Cv_{j_k}^k \\
&= (Cg)a_{\overline{k-1}|j_k} + C(1+g)v_{j_n}^n \\
&= (Cg)a_{\overline{k-1}|j_k} + Cv_{j_k}^{k-1} \cdot \frac{1+g}{1+j_k} \\
&< (Cg)a_{\overline{k-1}|j_k} + Cv_{j_k}^{k-1} = P_{k-1}(j_k)
\end{aligned}$$

where the last line follows from the fact that the bond is issued at discount, meaning $P < C$, therefore $g < j_k$ due to the price-discount formula for bonds: $P - C = C(g - j)a_{\overline{n}|j}$. Recall that $P_{k,j_k} = P_{k-1,j_{k-1}}$ due to how we define the yield rates. Therefore

$$P_{k-1}(j_{k-1}) < P_{k-1}(j_k)$$

Both sides represent a present value of positive cashflows, $P(j)$. Higher interest rates mean lower present value and vice-versa. So necessarily $j_{k-1} > j_k$. By definition of yield rate of a callable bond, we pick the lowest call data. The argument for premium is all reverse, since $g > j_k$. □

Bond Amortization Schedules

- We can think about the loan repayment scheme of the bond in terms of an amortization schedule, meaning we can deconstruct how much of the coupon and redemption payments are
- We track the development of "outstanding balance" of the debt in terms of a sequence called the **book values**, $\text{BV}_0, \text{BV}_1, ..., \text{BV}_m$. Note that BV_0 is the debt owed immediately after the bond is bought: this is the price paid for the bond. BV_t is the debt owed immediately after the t-th coupon is paid.

  Importantly, we have:

  - $\text{BV}_0 = P = C(g - j)a_{\overline{n}|j} + C$
  - $\text{BV}_m = C$, the redemption value.
  - Interpolating between these, we have: $\text{BV}_t = C(g - j)a_{\overline{n-t}|j} + C$
- The book values form a curve tracing from the original price of the bond to the redemption value.
- If the bond is premium, $P > C$, then the curve is decreasing, concave down. Therefore $B_t < B_{t-1}$, so $\text{PR}_t = B_{t-1} - B_t > 0$ (this is often written as P_t in the context of bonds, and called *adjustment of principal*). This is the standard loan repayment situation: part of the coupon pays for interest, and the rest pays for interest. Adjustment of principal for premium bonds is also referred to as **amortization of premium**.

  If the bond is discount, $P < C$, then the curve is increasing, concave up. Therefore, $B_t > B_{t-1}$, so $P_t = B_{t-1} - B_t < 0$. This means that the coupon Fr is insufficient to pay back the interest due, so the debt accumulates. In this case we call P_t the **accumulation of discount**.

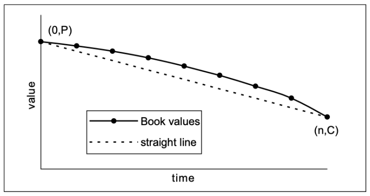

FIGURE (6.5.9) Balances of debt for a bond sold at a premium

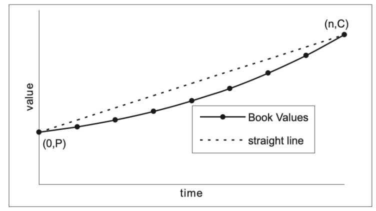

FIGURE (6.5.10) Balances of debt for a bond sold at a discount

Figure 1: Book value sequence for bond purchased at discount versis premium.

Observation 7. *Here is the amortization breakdown of an m-term bond.*

$$K_t = Fr = I_t + \mathrm{PR}_t$$
$$I_t = Fr - C(g-j)v_j^{m-t+1}$$
$$P_t = C(g-j)v_j^{m-t+1}$$

Proof. The key idea is that $I_t = jB_{t-1}$. Therefore

$$\begin{aligned} I_t = j \cdot \left(C(g-j)a_{\overline{m-(t-1)}|j} + C\right) &= C(g-j)(1-v^{t-m+1}) + Cj \\ &= Cg - Cj - C(g-j)(v^{t-m+1}) + Cj \\ &= Fr - C(g-j)(v^{t-m+1}) \end{aligned}$$

where we use $Cg = Fr$ in the last step. □

Term Structure of Interest Rates ("Yield Curve")

- Zero-risk, zero-coupon bonds (in practice, US treasury bonds play this role) are good measures of interest rates. There is no risk priced in. It is a very pure measure of the price of lending.

- From these secure bonds, we can extract spot rates. If $i_{[0,t]}$ is the yield rate of a bond maturing at t,
$$(1+s_t)^t := 1 + i_{[0,t]}$$

- From these spot rates, we can attempt to estimate $i_{[s,t]}$ in the natural way. We call this estimate the forward rate $f_{[s,t]}$ (referred to as the $(s-t)$-year forward rate deferred s years), and it is defined as follows for $s < t$.

$$(1+f_{[s,t]})^{t-s} := \frac{(1+r_t)^t}{(1+r_s)^s} = \frac{1+i_{[0,t]}}{1+i_{[0,s]}}$$

- The yield curve essentially plots the spot rate with respect to the term length. We typically think of the yield curve as increasing and hitting an asymptote, concave down. Flattening yield curve or inverted yield curve are signs of an unhealthy economy.

Durations

- In our study and application of interest, we make a lot of simplifying assumptions, most notably that interest rates are fixed. In this section we think carefully about the sensitivity of the present value of a cashflow to the interest rate. For simplicity, assume a flat yield curve, so we don't have to worry about different spot rates. Then the present value is a univariate function:

  $$P(i) = \sum_{t\geq 1} C_t v_i^t = \sum_{t\geq 1} C_t(1+i)^{-t}$$

  If we are using force of interest δ, then we have $P(\delta) = \sum_{t\geq 1} C_t e^{-\delta t}$.

  It is good to keep in mind how this kind of function is shaped. Note that $P(0) = \sum_t C_t$, and as i increases, this decays exponentially to zero.

- The **Macauley duration** can be thought of as a weighted sum of the actual duration until each cash flow, weighted by the present value of each one.

  $$\text{MacD}(i) := \frac{\sum_{t\geq 1} t \cdot C_t(1+i)^{-t}}{\sum_{t\geq 1} C_t(1+i)^{-t}}$$

  This captures sensitivity because cashflows farther out into the future are inherenty more unstable to changes in interest (since the interest is compounded over such a long period).

- The **modified duration** also measures sensitivity. You can think of it like a scaled derivative of the present value function with respect to the interest rate.

  $$\text{ModD}(i) := -P'(i)/P(i)$$

There is an important relationship between Macauley and modified duration.

Observation 8. $\text{ModD} = (1+i)\text{MacD}(i)$

Proof. Take the derivative.

$$P'(i) = \frac{\partial}{\partial i} \sum_{t \geq 1} C_t (1+i)^{-t} = \sum_{t \geq 1} -tC_t(1+i)^{-(t+1)} = -(i+1) \cdot \text{MacD}(i) \cdot P(i)$$

Rearranging, we have $-P'(i)/P(i) = (1+i)\text{MacD}(i)$. □

- **Modified** duration **first-order approximation**. This is a basic local linear approximation using the first derivative.

$$\begin{aligned} &P(i) \approx P(i_0)(1 - (i - i_0)\text{ModD}(i_0)) \\ \iff &P(i) \approx P(i_0) + (i - i_0) \cdot P'(i_0) \\ \iff &P'(i_0) \approx \frac{P(i) - P(i_0)}{i - i_0} \end{aligned}$$

  Personally I think it is best to start with the most intuitive thing to remember (like the last line) and work your way back to Modified Duration.

- **Macauley** duration **first-order approximation**.

$$P(i) \approx P(i_0)\Big(\frac{1+i_0}{1+i}\Big)^{\text{MacD}(i_0)}$$

Observation 9 (Macauley duration of level annuity-immediate)**.** *Let* $P(i) = v_i + v_i^2 + ... + v_i^n = a_{\overline{n}|i}$. *Then:*

$$\text{MacD}(i) = \frac{1v_i + 2v_i^2 + ... + nv_i^n}{v_i + v_i^2 + ... + v_i^n} = \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{1 - v^n}$$

Observation 10 (Macauley duration of bonds)**.** *Consider a par value bond with annual coupons* F*, coupon rate* r*, and annual yield* i*. Then*

$$\text{Mac}D(i) = \frac{Fr(Ia)_{\overline{n}|i} + nFv^n}{Fra_{\overline{n}|i} + Fv^n} = \frac{1+i}{i} - \frac{1+i+n(r-i)}{r((i+1)^n - 1) + i}$$

Portfolios

- Consider a portfolio which consists of some set of future cash flows (like debts to pay off; bond coupons or stock dividends to collect; obligations to buy or sell assets). We should be very familiar at this point with the idea that present value of this cash flow depends on the current interest rate and its development in the (near) future.

- The idea of immunization is to protect the present value of a portfolio against (small) changes to the interest rate.
- **Exact matching** is probably the most straightforward approach to immunization. This is done by ensuring that, at each time step, the value of the assets equals the value of the liabilities.
- **Redington immunization** is a set of conditions to ensure that the present value of a portfolio at . We say a cashflow is Redington immunized

  Condition 1: $P_A(i_0) = P_L(i_0)$ (PV of assets = PV of liabilities).

  Condition 2: $P'_A(i_0) = P'_L(i_0)$ (equiv., durations the same).

  Condition 3: $P''_A(i_0) > P''_L(i_0)$ (equiv., convexity of assets exceeds liabilities).

  In other words, if $f(i) = P_A(i_0) - P_L(i_0)$ then $f(i)$ is a local minimum.
- **Full immunization**: suppose we want to immunize against arbitrarily large fluctuations in the interest rate. This is a tall order, but we can come up with conditions.
  - (analytic) $f(i_0) = 0$ and is a global minimum.
  - (discrete-geometric) Suppose the cash flows are discrete. Then full immunization at i_0 occurs if $f(i_0) = f'(i_0) = 0$ and *every liability lies between two assets* ("liability sandwiching").

  By design, full immunization is a stronger condition than Redington
- To see how the discrete-geometric condition works, consider a cash flow with: asset A at time t, asset B_1 at time $t-u$, and asset B_2 at time $t+w$. Suppose we do things in terms of force of interest δ for simplicity. So

  $$f(\delta) = Ae^{-\delta t} - B_1e^{-\delta(t-u)} - B_2e^{-\delta(t+w)} = e^{-\delta t}(A - B_1e^{\delta u} - B_2e^{-\delta w})$$

  The first and second order conditions tell us:

  $$Ae^{-\delta_0 t} = B_1e^{-\delta_0(t-u)} + B_2e^{-\delta_0(t+w)}$$

  $$tAe^{-\delta_0 t} = (t-u)B_1e^{-\delta_0(t-u)} + (t+w)B_2e^{-\delta_0(t+w)}$$

  One can use these conditions to show $f(\delta) > 0$ for $\delta \neq \delta_0$.

- Note that exact matching does not imply Redington immunization. It ensures that the present values and the durations are the same but also that the convexities are the same.

- How should we think about the **difference between matching and immunization**? Here is a nice example adapted from [Vaaler et al., 2021] Chapter 9. Suppose you have a debt to pay off in 2 years. To pay off this debt you have three options: zero-coupon bonds maturing in 1, 2, and 3 years, respectively.

  If you buy purely 2-year bonds to cover this debt (the most straightforward thing), this is *exact matching.* You do not have to worry about changes in the interest rate because you've locked in a rate with this bond. The problem is that exact matching can be expensive in practice; maybe the 1 and 3 year bonds are much cheaper.

  If you buy purely 1-year bonds, you will hope for high interest rates in one year so you can reinvest for the second year and use this to pay off the debt.

  If you buy purely 3-year bonds, you will hope for low interest rates in two years, so that you can sell the bonds and use these to pay off the debt.

  If you buy a mix of 1 and 3 year bonds, perhaps you can be stable to changes in the interest rate. This is the basic idea of *immunization.*

Sources

Much of the material and notation is taken from [Vaaler et al., 2021]'s textbook on mathematical interest theory.

References

L. J. F. Vaaler, S. K. Harper, and J. W. Daniel. *Mathematical Interest Theory*, volume 57. American Mathematical Soc., 2021.