Coins, Partitions, and Generating Functions

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Outline

The Change-Making Problem

Problem (Change-making)

Say we have k cents. How many different ways can we express k cents as a combination of coins (penny, nickel, dime, quarter, etc.)?

Related questions:

- (Integer Knapsack problem) What is the minimum number of coins necessary?
- (Optimal Denomination) What series of coins minimizes the average number of necessary coins? (see Shallit 2002: "What this country needs is an 18 cent piece.")

Example

We can change 11 cents in four ways.

1. 11 pennies
2. 1 nickel, 6 pennies
3. 2 nickels, 1 penny
4. 1 dime, 1 penny (fewest coins)

(Note: greedy approach doesn't always give the fewest coins.)

Visualizing

k	0	5	10	15	20	25	30	35	40	45	50
p_k	1	1	1	1	1	1	1	1	1	1	1
n_k	1	2	3	4	5	6	7	8	9	10	11
d_k	1	2	4	6	9	12	16	20	25	30	36
q_k	1					13					49
h_k	1										50
a_k	1										

Mathematical Formulation

- We have n types of coins with values $\mathbf{w} = (w_1, w_2, ..., w_n)$.
- How many distinct $\mathbf{x} = (x_1, x_2, ..., x_n)$, with all $x_i \geq 0$, such that $\mathbf{x} \cdot \mathbf{w} = k$?

$$\sum_{i=1}^{n} x_i w_i = k$$

For instance, $(1, 5, 10, 25) \cdot (1, 0, 2, 1) = 46$

Solutions

Naive Recursion: Subtract each coin denomination, creating a tree. Treat each subproblem.

- $\approx O(m^n)$, where m is the number of coin types

Dynamic Programming: Build up solutions from 0 to n (or do recursion with record-keeping).

- $O(nm)$

Generating functions *can help us find an* $O(1)$ *solution!*

Motivation of Generating Functions

Recall the process of multiplying binomials (FOIL).

$$(1+x)(1+x) = 1 + x + x + x^2$$

Let $1 =$ "stay still" and $x =$ "take a step forward."
Frame it as a choice: "(stay or step) AND (stay or step)."
After this choice, there is:

1. 1 way to take 0 steps ($1x^0$)
2. 2 ways to take 1 step ($2x$)
3. 1 way to take 2 steps ($1x^2$)

Motivation

Consider the binomial theorem:

$$(1+x)^n = \sum_{k=1}^{n} \binom{n}{k} x^k$$

After n rounds, there are $\binom{n}{k}$ ways to have taken k steps.

The moral of the story is:
Polynomials can encode combinatorial information. (+ is "or", $*$ is "and")

Finite Change-Making

Say we had 1 penny, 2 nickels, and a dime. We can represent our problem as:

$$(1 + x)(1 + x^5 + x^{10})(1 + x^{10})$$

"(0 or 1 penny) AND (0 OR 1 OR 2 nickels) AND (0 or 1 dime)"

$$1 + x + x^5 + x^6 + 2x^{10} + 2x^{11} + x^{15} + x^{16} + x^{20} + x^{21}$$

This encodes that there are TWO ways to get 11 cents (if we start with this set of coins!).

The natural extension

Say we had INFINITE pennies, INFINITE nickels, and INFINITE dimes.
We can represent our problem as follows:

$$(1 + x + x^2 + ...)(1 + x^5 + x^{10} + ...)(1 + x^{10} + x^{20} + ...)$$

How many ways to represent k cents?
$\rightarrow$ **Find the coefficient of x^k.**

Definition

Definition (generating function)

The (ordinary) generating function of a sequence $(a_0, a_1, \ldots)$ is given by:

$$A(x) = \sum_{k=0}^{\infty} a_i x^k$$

A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag. (George Polya, *Mathematics and plausible reasoning* (1954)

A Solution?

We can very easily determine the generating function.

$$A(x) = (1+x+x^2+\ldots)(1+x^5+x^{10}+\ldots)(1+x^{10}+x^{20}+\ldots)(1+x^{25}+x^{50}+\ldots)(1+$$

Recall: $1 + x + x^2 + \ldots = \frac{1}{1-x}$

$$A(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})(1-x^{100})}$$

$\approx O(nm)$ runtime to get answer (Maclaurin series)

Towards constant-time solution

Since almost all the coin values are divisible by 5, we can shortcut.

$$B(x) = \frac{1}{(1-x)^2(1-x^2)(1-x^5)(1-x^{10})(1-x^{20})}$$

Conveniently, we can write:

$$A(x) = (1 + x + x^2 + x^3 + x^4)B(x^5)$$

$$b_k = a_{5k} = a_{5k+1} = a_{5k+2} = a_{5k+3} = a_{5k+4}$$

(Remember: a_n is our answer! So it suffices to know b_n.)

Solution (P1)

We can write $B(x)$ as follows

$$B(x) = \frac{C(x)}{(1 - x^6)^{20}}$$

where $C(x)$ is a (disgusting, but) finite polynomial.

Key: $\frac{1}{(1-x)^n} = \sum_{k=0}^{n} \binom{n+k-1}{n-1} x^k$

$$B(x) = C(x) \sum_{k=0}^{n} \binom{k+5}{5} x^{20k}$$

What is C?

$$
\begin{aligned}
C(x) &= \left(1 + x + \cdots + x^{19}\right)^2 \left(1 + x^2 + \cdots + x^{18}\right) \left(1 + x^5 + x^{10} + x^{15}\right) \\
&\quad \cdot \left(1 + x^{10}\right) \\
&= x^{81} + 2x^{80} + 4x^{79} + 6x^{78} + 9x^{77} + 13x^{76} + 18x^{75} + 24x^{74} + 31x^{73} \\
&\quad + 39x^{72} + 50x^{71} + 62x^{70} + 77x^{69} + 93x^{68} + 112x^{67} + 134x^{66} \\
&\quad + 159x^{65} + 187x^{64} + 218x^{63} + 252x^{62} + 287x^{61} + 325x^{60} + 364x^{59} \\
&\quad + 406x^{58} + 449x^{57} + 493x^{56} + 538x^{55} + 584x^{54} + 631x^{53} + 679x^{52} \\
&\quad + 722x^{51} + 766x^{50} + 805x^{49} + 845x^{48} + 880x^{47} + 910x^{46} + 935x^{45} \\
&\quad + 955x^{44} + 970x^{43} + 980x^{42} + 985x^{41} + 985x^{40} + 980x^{39} + 970x^{38} \\
&\quad + 955x^{37} + 935x^{36} + 910x^{35} + 880x^{34} + 845x^{33} + 805x^{32} + 766x^{31} \\
&\quad + 722x^{30} + 679x^{29} + 631x^{28} + 584x^{27} + 538x^{26} + 493x^{25} + 449x^{24} \\
&\quad + 406x^{23} + 364x^{22} + 325x^{21} + 287x^{20} + 252x^{19} + 218x^{18} + 187x^{17} \\
&\quad + 159x^{16} + 134x^{15} + 112x^{14} + 93x^{13} + 77x^{12} + 62x^{11} + 50x^{10} \\
&\quad + 39x^9 + 31x^8 + 24x^7 + 18x^6 + 13x^5 + 9x^4 + 6x^3 + 4x^2 + 2x + 1.
\end{aligned}
$$

$$C(x) = c_0 + c_1 x + \ldots + c_{80} x^{80}$$

Solution (P2)

$$B(x) = \left[\sum_{j=0}^{80} c_j x^j\right]\left[\sum_{k=0}^{n} \binom{k+5}{5} x^{20k}\right]$$

We want a_n. Suffices to find b_n. Interested in j, k such that:

$$j + 20k = n$$

$$\implies n \equiv j \mod 20$$

There are at most five such j.
Thus, the solution n is a sum of at most 5 terms. □

Broader Question

Suppose we had a one-cent coin, a two-cent coin, a three-cent coin, etc. Then our total generating function would become:

$$A(x) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}$$

This sort of change-making is equivalent to the idea of **partitioning**. The function above is the **partition generating function**.

Definitions

Definition (partition)

A partition of $n \in \mathbb{Z}_{>0}$ is a k-tuple of positive integers $(a_1, ..., a_k)$, arranged in descending order, such that

$$\sum_{i=1}^{k} a_i = n$$

The total number of distinct partitions of n objects into sets of size k is denoted by $p_{n,k}$.

Intuitively: how many ways can I arrange n objects into k distinct piles?

Illustration

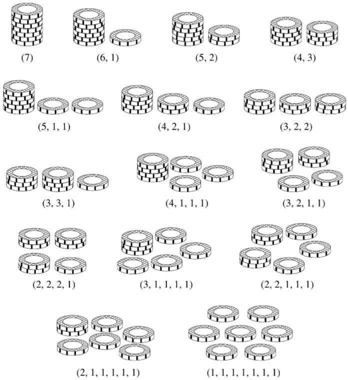
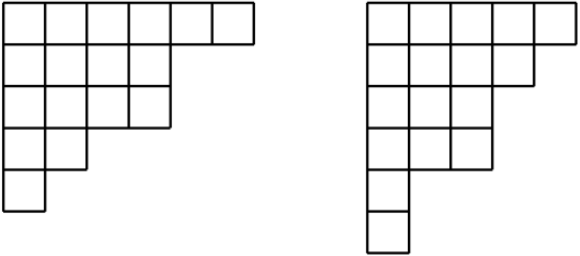


FIGURE 2.13. The fifteen ways to stack seven poker chips.

Young Diagram



(a) $\lambda = (6, 4, 4, 2, 1)$. (b) $\lambda' = (5, 4, 3, 3, 1, 1)$.

FIGURE 2.14. The Young diagram for a partition λ, and its conjugate λ'.

Definitions

Definition (partition function)

The total number of partitions one can generate from n objects is, naturally, given by:

$$p_n = \sum_{k=1}^{n} p_{n,k}$$

The first few values of the partition function:

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490\ldots$$

Recursive formula

Given any partition of n, denoted $(a_1, \ldots, a_k)$, we know:

1. If $a_k = 1$, then $(a_1, \ldots, a_{k-1})$ is a partition of $n-1$. The number of such partitions that end with 1 is therefore $p_{n-1,k-1}$.
2. If $a_k \neq 1$, then $(a_1 - 1, \ldots, a_k - 1)$ is a partition of $n-k$, so there are precisely $p_{n-k,k}$ partitions that don't end in 1.

This allows us to construct a recursive formula for partitions:

$$p_{n,k} = p_{n-1,k-1} + p_{n-k,k}$$

Euler Pentagonal Number Theorem

The Euler function $\phi(x)$ is the inverse of the partition generating function.

$$\phi(x) = \prod_{k}^{\infty}(1 - x^k) = \sum_{-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2}$$

Note that the pentagonal numbers are of the form $(3n^2 - n)/2$.

$$1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, \ldots$$

Euler Theorem Result

In particular, this implies:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$$

$$p(n) = \sum_{k \neq 0}^{\infty} (-1)^k p(n - g_k)$$

This is a faster way of computing the function.

Hardy-Ramanujan

The partition function converges quite nicely, as $n \to \infty$.

$$p_n \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

Asymptotic formula obtained by G. H. Hardy and Ramanujan in 1918. Kind of weird.