q-analogs

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1 Abstract

In this talk, we will introduce the subject of q-analogs, which generalize combinatorical objects into functions of a formal variable q (and recover the original concept when we send $q \to 1$). We prove that the q-analog for $n!$ provides us with the generating function for inversions, and we explore basic results on the q-binomial coefficient. With time permitting, we discuss where q-analogs come up in the study of symmetric functions.

2 Introduction

The story of q-analogs begins, with some sense, in the following elementary calculus exercise.

$$\lim_{q \to 1} \frac{q^n - 1}{q - 1}$$

If you apply L'Hopital's rule, this comes out to n, and you can simply move on with your life. But you could also take an imaginative leap, and ask: what if we took away the limit? What we track an association between n and this (somewhat elegant) expression $\frac{q^n-1}{q-1}$. Let us see where this takes us.

Definition (q-numbers)

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + ... + q^{n-1}$$

We use the following notation to describe a factorial on this number:

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$$

Recall that $n!$ counts the total number of permutations of length n. $[n]_q!$ does as well (we just take $q \to 1$, as per the motivating identity). However, in its pure form, $[n]_q!$ is a polynomial. So setting $q \to 1$ effectively leads us to add the coefficients.

We are losing data in this process: namely, we are losing track of the coefficients and the exponents. We will find out that such data is actually quite interesting. But first, a definition.

Definition: The **inversion number** of a permutation $\pi = \pi_1\pi_2...\pi_n$ is, morally speaking, the number of elements that are out of order. More formally, we write:

$$\text{inv}\pi = \#\{(i,j) : i < j, \pi(i) > \pi(j)\} = \#\text{Inv}\pi$$

Note that the inversion set $\#\text{Inv}\pi$ counts the indices which are out of order, not the elements.

- $(\pi_1 = 3, \pi_2 = 1, \pi_3 = 2))$ has inversion set containing $(1,2), (1,3)$. The inversion number is thus 2.
- There are 6 elements in S_3. Only $1, (1,2), (2,3)$, and $(1,2,3)$ have inversion number 0.

Some wishful thinking and some careful algebra with examples might lead us to the following statement, which is in fact an important result in this classical treatment of q-analogs.

Theorem: For $n \geq 0$, we have the following identity:

$$\sum_{\pi \in S_n} q^{\text{inv}\pi} = [n]_q!$$

Proof: We prove by induction on n. The base case $n = 1$ gives us a trivial equality. For the inductive step, suppose the statement is true for $n - 1$.

Observe that any permutation $\pi \in S_n$ written as a list of numbers can be uniquely obtained by $\sigma \in S_{n-1}$ by inserting n into one of the n spaces between elements of σ (and including the edges).

Let σ^i be the result of placing n in the i-th spot (counting from the right). Then clearly we should have inv$(\sigma^i) = i +$ inv(σ). This is because the term we placed at the i-th position is the largest, so the i terms after it will contribute to the inversion number. For instance, in the aforementioned example of 312, if we place 4 in the $i = 1$ position, we get 3142 and the inversion number increases by $i = 1$.

$$\sum_{\pi \in S_n} q^{\text{inv}\pi} = \sum_{\pi \in S_{n-1}} \sum_{i=0}^{n-1} q^{\text{inv}\sigma^i} = \sum_{\pi \in S_{n-1}} \sum_{i=0}^{n-1} q^{i+\text{inv}\sigma} = \sum_{\pi \in S_{n-1}} q^{\text{inv}\sigma} \sum_{i=0}^{n-1} q^i$$

By the inductive hypothesis, the first sum is $[n-1]_q!$. By definition, the second term is $1 + q + q^2 + ... + q^{n-1} = [n]_q$. So we end up with the desired expression and the proof is complete. $\square$

We can couch the theorem in the following terms: for a given symmetric group S_n, we can keep track of a **statistic** called the inversion number. We can ask ourselves what the frequency distribution of inversion numbers look like, i.e. the number of permutations in S_n with k inversions. We can convert this into a generating function (which, of course, should have a finite number of terms). The q-factorial function provides us quite neatly with this generating function.

3 The q-binomial coefficient

It might seem natural to ask whether this q-analogy extends from factorials to binomial coefficients. Let's see.

Definition (q-binomial coefficient): Defined exactly as you would expect (with the caveat that if $k = 0$, you only get only get 1 if $n = 0$ as well; otherwise, you get zero because you cannot pick nothing from something...).

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$$

You might wonder whether this actually yields a polynomial rather than a rational function. We can show it does the former when we show its inductive formula.

Theorem (Recursive formulae for q-binomial coefficient): Take $n \geq 0$. We have the following identities.

$$\binom{0}{k}_q = \mathbf{1}[k = 0]$$

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$$

Proof: All of these follow directly from the definitions. We will show the third identity; the first and second are left as exercises.

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n-1]_q!}{[k]_q![n-k]_q!}[n]_q$$

We claim that $[n]_q = [k]_q + q^k[n-k]_q$. We explain this as follows: the $[k]_q!$ accounts for the first k terms, and the $[n-k]_q!$ accounts for the last $n-k$ terms;

the q^k is needed to shift the terms correctly (the sum would need to start at q^k rather than 1). Thus, we have:

$$\frac{[n-1]_q!}{[k]_q![n-k]_q!}([k]_q + q^k[n-k]_q)$$

$$\frac{[n-1]_q!}{[k-1]_q![n-k]_q!} + q^k\frac{[n-1]_q!}{[k]_q![n-k-1]_q!} = \binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q \quad \square$$

We are now ready to prove the q-analogy to a very important theorem.

$$1$$

$$1 \qquad\qquad 1$$

$$1 \qquad\qquad 1+q \qquad\qquad 1$$

$$1 \qquad 1+q+q^2 \qquad\qquad 1+q+q^2 \qquad 1$$

$$1 \quad 1+q+q^2+q^3 \quad 1+q+2q^2+q^3+q^4 \quad 1+q+q^2+q^3 \quad 1$$

Figure 3.7. The top of the q-Pascal triangle of q-binomial coefficients

Figure 1: Illustration of the recursive formula, in terms of a Pascal's Triangle.

Theorem (q-binomial theorem.): For any positive integer n,

$$(1+qt)(1+q^2t)\cdots(1+q^{n-1}t) = \sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}t^k$$

Proof: Induct on n. Base case $n = 0$ is trivial. For the inductive step, apply the formula for the q-binomial coefficient and simplify.

$$\sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}}t^k = \sum_{k=0}^{n}\binom{n-1}{k}_q q^{\binom{k}{2}}t^k + q^{n-k}\sum_{k=0}^{n}\binom{n-1}{k-1}_q q^{\binom{k}{2}}t^k$$

If the first sum is denoted S and the second sum is denoted T, then it suffices to show that $T = q^{n-1}tS$. This is because, by the inductive hypothesis, $S = (1+qt)(1+q^2t)\cdots(1+q^{n-2}t)$, so $S(1+(q^{n-1}t))$ is precisely the desired term.

Let us show $T = q^{k-1}tS$. This follows from the observation that:

$$q^{n-k}\sum_{k=0}^{n}\binom{n-1}{k-1}_q q^{\binom{k}{2}}t^k = q^{n-1}t\sum_{k=0}^{n}\binom{n-1}{k-1}_q q^{\binom{k}{2}-k+1}t^{k-1}$$

We have the basic identity $\binom{k}{2} - k + 1 = \binom{k-1}{2}$. Thus, if do a simple substitution in the sum with $k' = k-1$, then we have the sum becoming S and thus:

$$q^{n-k}\sum_{k=0}^{n}\binom{n-1}{k-1}_q q^{\binom{k}{2}}t^k = q^{n-1}tS \quad \square$$

4 Examples

The q-binomial coefficient shows up in a lot of places. They help us count, for instance, (1) vector spaces over finite fields, (2) the sizes of partitions contained within a given Young's Diagram, and (3) the area under a lattice path. We will discuss (3) in more depth.

4.1 Lattice Paths

Recall that the number of paths from the origin to the point (m, n), using only northerly and westerly steps, is given by $\binom{m+n}{m}$. We call these lattice paths.

For a given lattice path, we can ask ourselves about the area below it. We draw pictures to visualize this. Let $A(p)$ be the area associated with a given lattice path.

Theorem: Let $\mathcal{P}$ be the set of lattice paths from $(0, 0)$ to (m, n).

$$\sum_{p\in\mathcal{P}} q^{A(p)} = \binom{m+n}{m}_q$$

Proof: If we call the left hand size $F(m, n)$, we should note $F(0, n) = F(m, 0) = 0$. For nontrivial $F(m, n)$ we have a neat way of visualizing:

- If the last step was northerly, then the area of whatever path we end up with was the same as the area we had with the path that ended at $(m, n - 1)$.
- If the last step was easterly, then this path was a path to $(m - 1, n)$ followed by a step which contributed n to the area.

This leaves us with the recursion which, with these initial conditions, equates $F(m, n)$ with the q-binomial coefficient. Observe that the q^n signifies how much area we add to the paths from the second case,

$$F(m, n) = F(m, n - 1) + q^n F(m - 1, n) \quad \square$$

This works because we have the correspondence $F(m, n) \mapsto \binom{m+n}{n}_q$.

5 Curiosities

In case you are interested in other kinds of q-analogs, we present some below.

q-derivative:

$$\left(\frac{d}{dx}\right)_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

q-Pochhammer symbol:

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$$

q-Vandermonde convolution: This is the analogy to a usual formula involving the binomial coefficients. Observe the discrete convolution that is occurring.

$$\binom{m+n}{r} = \sum_{k=0}^{n} \binom{m}{k} \binom{n}{r-k}$$

With the q-analogues, we have:

$$\binom{m+n}{r}_q = \sum_{k=0}^{n} \binom{m}{k}_q \binom{n}{r-k}_q q^{j(m-k+j)}$$

6 Sources

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