$\boxed{\text{TALK}}$ **Outline**: ① Euler Pentagonal Number Theorem

② q-hypergeometric series $F(a,b;t)$

③ Iteration $F(a,b;t) \longrightarrow F(a,bq;t)$

"$b \longrightarrow bq$"

④ Special case $\Rightarrow$ proof of PNT

PNT
$$\prod_{n\geq 1}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}$$

$$(1-q)(1-q^2)(1-q^3)\cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15}$$

Remark 1:
$$\frac{1}{\prod_{n\geq 1}(1-q)^n} = \sum_{n\geq 0} p(n)\, q^n$$
where $p(n)$ is the partition func.

This gives a great recursive formula for the partition function:
$$1 = \left(\sum_{-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}\right)\left(\sum_{n\geq 0} p(n)\, q^n\right)$$

$$\cancel{p(5)} \Rightarrow a_0 p_0 = 1 \qquad \underbrace{\sum_{i=0}^{n} p(n-i)\, a_i}_{\text{coefficient of degree } n} = 0$$

~~Def a_n =~~

$$\text{where} \quad a_i = \begin{cases} 1 & \text{if } i = \frac{1}{2}(3k^2 \pm k),\ k \text{ even} \\ -1 & \text{if } i = \frac{1}{2}(3k^2 \pm k),\ k \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

For instance $\quad p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7)$

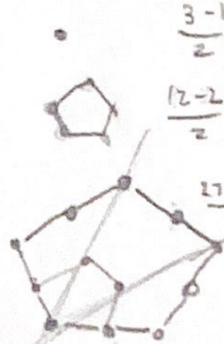
<u>Rmk 2</u>: Why "pentagonal"?

For $k \in \mathbb{N}$: $(3k^2-k)/2$ corresponds to the # of dots in a pentagon diagram,

To derive, consider:

$$q_n = q_{n-1} + 3n - 2$$

(with each step, we add $3n-2$ vertices)



	$\frac{3k^2-k}{2}$	q_n
$k=1$	$\frac{3-1}{2}$	1
$k=2$	$\frac{12-2}{2}$	5
$k=3$	$\frac{27-5}{2}$	12

We extend the concept for $k \in \mathbb{Z}$: $\underset{k \geq 0}{\frac{3k^2-k}{2}}$, $\underset{k<0}{\frac{3k^2+k}{2}}$

n	-3	-2	-1	0	1	2	3
q_n	15	7	2	0	1	5	12

We want to introduce a tool that will allow us to prove Euler's pentagonal number theorem relatively easily.

$\boxed{q\text{-hypergeometric series}}$

Recall regular hypergeometric series

$c_0, c_1, c_2, \ldots, c_n, \ldots$ s.t. $\dfrac{c_{n+1}}{c_n} = \dfrac{P(n)}{Q(n)} = \dfrac{(n-a_1)\cdots(n-a_p)}{(n-b_1)\cdots(n-b_q)} t$

$$c_n = \frac{c_n}{c_{n-1}} \frac{c_{n-1}}{c_{n-2}} \cdots \frac{c_1}{c_0} = \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} t^n \qquad (a)_n = a(a+1)\cdots(a+n-1)$$

"shifted factorial"

$$\boxed{{}_pF_q\left(\begin{matrix} a_1 \cdots a_p \\ b_1 \cdots b_q \end{matrix}\; z\right) = \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} t^n}$$

We pursue a q-analog:

$$\boxed{{}_aF_b(t,q) = 1 + \sum_{n \geq 1} \frac{(1-aq)(1-aq^2)\cdots(1-aq^n)}{(1-bq)(1-bq^2)\cdots(1-bq^n)} t^n}$$

"q-shifted factorial": $(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$

$$(a)_\infty = \prod_{n \geq 0}(1-aq^n)$$

$$\left(\text{Note that } (q)_\infty = \prod_{n \geq 1}(1-q^n) \overset{\text{Euler's Pent}}{=} \sum_{-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2}\right)$$

Abbrev:

$$\boxed{F(a,b;t;q) = \sum_{n \geq 0} \frac{(aq)_n}{(bq)_n} t^n} = F(a,b;t)$$

(q-implicit)

...ing Transformations

There are nice formulas that allow us to go from

$$F(a,b;t) \xrightarrow{(i)} F(aq;bq;t) \qquad (a,b)\to(aq,bq)$$

$$\xrightarrow{(ii)} F(a,b;tq) \qquad (t)\to(tq)$$

$$\xrightarrow{(iii)} F(a,bq;t) \qquad (b)\to(bq)$$

(*) Main one of interest

Easy to show (i): since $(aq)_n = (1-aq)(aq^2)_{n-1}$

then $$F(a,b;t) = 1+\sum_{n\geq 1}\frac{(aq)_n}{(bq)_n}t^n = 1+\sum_{n\geq 1}\frac{(1-aq)}{(1-bq)}\frac{(aq^2)}{(bq^2)}t^n$$

$$= 1+\frac{(1-aq)}{(1-bq)}tF(aq,bq;t)$$

(ii) Let $A_n = \frac{(aq)_n}{(bq)_n}$ $\qquad (1-bq^{n+1})A_{n+1} = (1-aq^{n+1})A_n$

$$\sum_{n\geq 0}A_{n+1}t^{n+1} - b\sum_{n\geq 0}A_{n+1}(tq)^{n+1}$$

$$= t\sum_{n\geq 0}A_n t^n - atq\sum_{n\geq 0}A_n(tq)^n$$

$$\Rightarrow F(a,b;t) = \frac{1-b}{1-t} + \frac{b-atq}{1-t}F(a,b;tq)$$

(i) + (ii) $\Rightarrow$
$$F(a,b;t) = \frac{1-atq}{1-t} + \frac{(1-aq)(b-atq)}{(1-bq)(1-t)} tq\, F(aq, bq; tq)$$

$$\circledast \quad (1-t)F(a,b;t) = 1 + \sum_{n\geq 1} (A_n - A_{n-1}) t^n$$

Recall $A_0 = 1$ and $A_{n-1} = \frac{1-bq^n}{1-aq^n} A_n$

Then: $A_n - A_{n-1} = (b-a) q^n \frac{A_n}{1-aq^n}$

$$\circledast \quad (1-t) F(a,b;t) = 1 + (b-a) \sum_{n\geq 1} \frac{(aq)_{n-1}}{(bq)_n} (tq)^n$$

$$= 1 + \frac{b-a}{1-bq} \sum_{n\geq 1} \frac{(aq)_{n-1}}{(bq)_{n-1}} (tq)^n$$

$$\Rightarrow F(a,b;t) = \frac{1}{1-t} + \frac{(b-a)tq}{(1-bq)(1-t)} F(a, bq; tq)$$

Applying $t \to tq$ in reverse, we get:

$$\text{(iii)} \quad \boxed{F(a,b;t) = \frac{b}{b-at} + \frac{(b-a)t}{(1-bq)(b-at)} F(a, bq; t)}$$

$b \to bq$

plan is to iterate $b \to bq$

$f_0 = F(a, b; t) \qquad f_n = F(a, bq^n; t)$

Then let $n \to \infty$, $f_n \to S = F(a, 0; t)$

$$f_n = \underbrace{\frac{bq^n}{bq^n - at}}_{L_n} + \underbrace{\frac{(bq^n - a)t}{(1 - bq^{n+1})(bq^n - at)}}_{M_n} f_{n+1}$$

Ask more general question: how to find f_0 when we have $f_n = L_n + M_n f_{n+1}$

$$\begin{aligned} f_0 = L_0 + M_0 f_1 &= L_0 + M_0 (L_1 + M_1 f_2) \\ &= L_0 + M_0 (L_1 + M_1 (L_2 + M_2 f_3)) \\ &= \underbrace{L_0 + M_0 L_1 + M_0 M_1 L_2}_{G_2} + \underbrace{M_0 M_1 M_2}_{H_2} f_3 \end{aligned}$$

In general, $H_N = M_0 M_1 \cdots M_N$

$$G_N = L_0 + \sum_{r=0}^{N-1} L_{r+1} (M_0 M_1 \cdots M_r)$$

$$f_0 = G_N H_N f_{N+1}$$

If, miraculously, $f_N \to f$
$G_N \to G \quad \Rightarrow \quad f_0 = G + Hf$
$H_N \to H$

Returning to iteration of $b \to bq$

$$L_n = \frac{bq^n}{bq^n - at} \qquad M_n = \frac{(bq^n - a)t}{(1 - bq^{n+1})(bq^n - at)}$$

$$H_n = \prod_{r=0}^{n} \frac{(1 - (b/a)q^r)}{(1 - bq^{r+1})(1 - (b/at)q^r)}$$

$$G_n = \frac{-(b/at)}{1 - (b/at)} \sum_{k=0}^{n} \frac{(b/a)_k}{(bq)_k (bq/at)_k} q^k$$

$$f_n \to \underset{S}{\underset{\parallel}{F(a, 0; t)}} \quad (\text{for } |q| < 1)$$

$$H = \frac{(b/a)_\infty}{(bq)_\infty (b/at)_\infty} \qquad G = \frac{(b/at)}{1 - (b/at)} \sum_{n \geq 0} \frac{(b/a)_n}{(bq)_n (bq/at)_n} q^n$$

$$\boxed{f_0 = F(a, b; t) = HS - G}$$

Special case of $F = HS - G$

$a \equiv b^2/t$.

Then:

$$\begin{cases} S = F(b^2/t; 0; t) = \sum_{n\geq 1} \frac{(-1)^n b^{2n} q^{(n^2+n)/2}}{(t)_{n+1}} \\ H = (t/b)_\infty \big/ (bq)_\infty (b^{-1})_\infty \\ G = \frac{b^{-1}}{1-b^{-1}} \sum_{n\geq 1} \frac{(t/b)_n}{(bq)_n (b^{-1}q)_n} q^n \\ F = F(b^2/t; b; t) \end{cases}$$

Let $t \to 0$

Then

$$\begin{cases} S = \sum_{n\geq 0} (-1)^n b^{2n} q^{(n^2+n)/2} \\ H = (qb)^{-1}_\infty (b^{-1})^{-1}_\infty \\ G = -\frac{1}{1-b} \sum_{n\geq 0} \frac{q^n}{(bq)_n (b^{-1}q)_n} \\ F = (1-b) F(b, 0; b) \end{cases}$$

Power series expansion for $h(b) = (1-b) F(b, 0; b)$
(applying $(a, b, t) \to (qa, b, qt)$)

$$h(b) = 1 - b^2 q - b^3 q^2 h(bq)$$

Substitute power series for $h(b)$, equate coeffs:

$$h(b) = 1 - b^2 q - b^3 q^2 h(bq)$$

$$\Downarrow$$

$$(1-b)\,F(b,0;b) = \sum_{n\geq 0} (-1)^n q^{(3n^2+n)/2} b^{3n} + \sum_{n\geq 0} (-1)^n q^{(3n^2-n)/2} b^{3n-1}$$

Let $b \to 1$. Then

$$\lim_{b\to 1} (1-b)F(b,0;b) = \prod_{-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} \qquad \square$$

$$\| $$

$$\prod_{n\geq 1} (1-q^n) = (q)_\infty$$

Why? As a result of $t \to tq$:

$$\lim_{t\to 1} (1-t)\,F(a,b;t) = (1-b)\,F(a/b,1;b) = \frac{(aq)_\infty}{(bq)_\infty}$$

$$\cancel{\lim_{b\to 1} (1-b)\,F(b,0;b) =}$$

For $a=b$, $b=0$, $t=b$ $\Rightarrow (bq)_\infty$

$$\downarrow{\scriptstyle b\to 1}$$

$$(q)_\infty \qquad \square$$

How to prove limit identity

$L_n = \frac{1-b}{1-tq^n}$ "$f_n = L_n + M_n f_{n+1}$"

$t \to tq$ $f_0 = F(a,b;t)$

$f_n = F(a,b;tq^n)$

$M_n = b - atq^n$

$L_n \to 1-b$

$M_n \to b$

$f_n \to F(a,b;0) = 1$

$$H_n = \prod_{r=0}^{n} \frac{(b-atq^{r+1})}{(1-tq^r)}$$

$\boxed{\text{Cases}}$ $b=0$, $b=1$, $0<|b|<1$ $H_n \to 0$

$$\Rightarrow F(a,b;t) = \frac{1-b}{1-t} F\left(\frac{at}{b}, t; b\right)$$

Hence, $(1-t)F(a,b;t)$ invariant under involution $a' = at/b$, $b' = t$, $t' = b$

$$\lim_{t\to 1} (1-t) F(a,b;t) =$$

$$(1-b) F(a/b, 1; b) = \frac{(aq)_\infty}{(bq)_\infty} \quad \square$$

(could have been proven directly, but this is quite elegant

Other nice identities

$$\sum_{n\geq 0} q^{(n^2+n)/2} = \prod_{n\geq 1} \frac{(1-q^{2n})}{(1-q^{2n-1})} \qquad \text{(Gauss)}$$

$$\sum_{-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n\geq 1} \left(\frac{1-q^n}{1+q^n}\right) = \prod_{n\geq 1} (1-q^{2n})(1-q^{2n-1})^2$$

$$\sum_{-\infty}^{\infty} q^{n^2} = \prod_{n\geq 1} (1-q^{2n})(1+q^{2n-1})^2$$

Elliptic theta functions, special case of Jacobi identity.