The Duality of Trace and Determinant

Noah Bergam

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Outline

Tensors

The tensor product $V \otimes W$ consists of *expressions* of the form:

$$v_1 \otimes w_1 + v_2 \otimes w_2 + \ldots + v_n \otimes w_n$$

This $\otimes$ is multi-linear, i.e. linear in each component.

$$(\lambda_1 v_1 + v_2) \otimes w_1 = \lambda_1 (v_1 \otimes w_1) + (v_2 \otimes w_1) \tag{1}$$

Contextualizing Tensors

- **Cartesian Product** $V \times W$: tuples of the underlying set.
- **Direct Product** $V \times W$: tuples as a vector space, coordinate-wise.
- **Direct Sum** $V \oplus W$: tuples with finitely many nonzero components.
- **Tensor Product** $V \otimes W$: tuples with multi-linearity
- **Wedge Product** $V \wedge V$: tuples with multi-linearity and anti-symmetry.

(Note: the difference between direct sum and direct product only emerges for infinite sums/products.)
(Note: you cannot wedge different vector spaces, since this would make anti-commutativity ill-defined.)

Exterior Power

Definition (Exterior Power)

For a vector space V, the n-th exterior power of V, denoted $\Lambda^n V$, is spanned by elements of the following form for $v_1, ..., v_n \in V$.

$$v_1 \wedge ... \wedge v_n$$

which obey the multi-linearity and anti-symmetry. For example:

$$(\lambda v_1 + w) \wedge ... \wedge v_n = \lambda(v_1 \wedge ... \wedge v_n) + (w \wedge ... \wedge v_n)$$

$$v_1 \wedge v_2 \wedge ... \wedge v_n = -(v_2 \wedge v_1 \wedge ... \wedge v_n)$$

Notes on Exterior Product

Important Properties:

1. (vanishing) $v \wedge v = 0$
2. (associativity) $(v \wedge w) \wedge x = v \wedge (w \wedge x)$
3. (untangling) $v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(n)} = \mathrm{sgn}(\sigma)(v_1 \wedge \ldots \wedge v_n)$

Warning: Not every element in $\Lambda^n V$ is *reducible* to a single wedge product (in general, it is a linear combination of such elements).

The k-linear extension

Definition

For $A \in \mathrm{End}(V)$, the k-linear extension $\Lambda^N A^k : \Lambda^N V \mapsto \Lambda^N V$ defined as:

$$\Lambda^m A^k \left(\bigwedge_{j=1}^{m} v_j \right) = \sum_s \bigwedge_{j=1}^{m} A^{s_j} v_j$$

$$\text{where} \qquad s \in \{0,1\}^n \qquad \sum_j s_j = 1$$

$\Lambda^N A^N$ means we apply A to each entry (one-dimensional).
$\Lambda^N A^1$ means we apply A to only one entry (n-dimensional).

The k-linear extension (examples)

For $\Lambda^N A^k$ I am applying A to k entries of the wedge product.

$$\Lambda^3 A^1(v_1 \wedge v_2 \wedge v_3) = Av_1 \wedge v_2 \wedge v_3 + v_1 \wedge Av_2 \wedge v_3 + v_1 \wedge v_2 \wedge Av_3$$

If V is N-dimensional, then $\dim(\Lambda^N A^k) = \binom{N}{k}$.

$$\begin{aligned} \Lambda^3 A^2(v_1 \wedge v_2 \wedge v_3) &= Av_1 \wedge Av_2 \wedge v_3 \\ &+ Av_1 \wedge v_2 \wedge Av_3 \\ &+ v_1 \wedge Av_2 \wedge Av_3 \end{aligned}$$

Determinant and Trace

Definition

The **determinant** of $A \in \mathrm{End}(V)$ is the number by which any nonzero tensor $\omega \in \Lambda^N V$ is multiplied when $\Lambda^N A^N : \Lambda^N V \mapsto \Lambda^N V$ acts on it.

$$(\Lambda^N A^N)\omega = (\det A)\omega$$

Definition

The **trace** of $A \in \mathrm{End}(V)$ is the number by which any nonzero tensor $\omega \in \Lambda^N V$ is multiplied when $\Lambda^N A^1 : \Lambda^N \mapsto \Lambda^N$ acts on it.

$$(\Lambda^N A^1)\omega = (\mathrm{tr}\ A)\omega$$

Note: For $\dim(V) = N$, $\Lambda^N V$ is one-dimensional. $\Lambda^1 V$ is n-dimensional.

Illustration of wedge-based determinant

$$\Lambda^n A^n \omega = \Lambda^n A^n (v_1 \wedge \dots \wedge v_n) = (Av_1 \wedge \dots \wedge Av_n)$$

$$= \left(\sum_{j_1=1}^{n} A_{j_1,1} v_{j_1} \wedge \dots \wedge \sum_{j_1=1}^{n} A_{j_n,n} v_{j_n} \right)$$

$$= \sum_{j_1=1}^{n} \dots \sum_{j_n=1}^{n} \left(A_{j_1,1} v_{j_1} \wedge \dots \wedge A_{j_n,n} v_{j_n} \right)$$

$$= \sum_{j_1=1}^{n} \dots \sum_{j_n=1}^{n} (A_{j_1,1} \cdots A_{j_n,n}) \left(v_{j_1} \wedge \dots \wedge v_{j_n} \right)$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} A_{j,\sigma(j)} (v_1 \wedge \dots \wedge v_n) = \det(A)\omega \quad \square$$

Illustration of wedge-based trace

$$\Lambda^n A\omega = \Lambda^n A(v_1 \wedge \ldots \wedge v_n) = \sum_{i=1}^{n} v_1 \wedge \ldots \wedge (Av_i) \wedge \ldots \wedge v_n$$

$$= \sum_{i=1}^{n} \sum_{j_1=1}^{n} A_{j_i,i}(v_1 \wedge \ldots \wedge v_{j_i} \wedge \ldots \wedge v_n)$$

This is zero unless $v_{j_i} = i$. This eliminates the second sum and recovers the usual formula.

$$= \sum_{i=1}^{n} A_{ii}(v_1 \wedge \ldots \wedge v_n) = \mathrm{tr}(A)\omega$$

Theorems

Theorem (Liouville's Formula)

Let $A \in End(V)$.

$$\det(\exp(A)) = \exp(tr(A))$$

where $\exp(A) = 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^2 + \ldots$ *denotes the matrix exponential. More generally,* $\det(\exp(t \cdot A(t))) = \exp(t \cdot tr(A(t)))$*, where t is a formal variable and $A(t)$ is an operator-valued formal power series.*

Theorem (Jacobi's Formula)

For $F(t)$ an operator-valued formal power series such that $F^{-1}(t)$ exists:

$$\partial_t \det(F(t)) = (\det F(t)) \; tr(F^{-1}(t) \cdot \partial_t F(t))$$

Context and Game Plan

An **operator-valued formal power series** is just a function $F(t) = 1 + F_1 t + F_2 t^2 + \ldots$ where the coefficients $F_i \in \mathrm{End}(V)$. This is more flexible than just talking about linear operators.

The idea is to represent both $\det(\exp(A(t)))$ and $\exp(\mathrm{tr}(A(t)))$ as a (formal) power series in t satisfying some differential equation.

- First we establish some theory on how to solve differential equations for formal power series.
- Then we will guess a suitable differential equation that will enable us to prove the identity.

Characterization of $\exp(tA)$

Lemma

The operator-valued function $F(t) = \exp(tA)$ is the unique solution to the following differential equation.

$$\partial_t F(t) = F(t)A \qquad F(0) = 1_V$$

Proof of Characterization of $\exp(tA)$

Lemma

For $A \in End(V)$. the operator-valued function $F(t) = \exp(tA)$ is the unique solution to the following differential equation.

$$\partial_t F(t) = F(t)A \qquad F(0) = 1_V$$

Proof.

Since $F(0) = 1$, we know $F(t) = 1 + F_1 t + F_2 t^2 + \ldots$
Note $F'(0) = A = F_1 A$, $F''(0) = A^2 = 2F_2$, $F'''(0) = A^3 = 6F_2$, etc.
Matching coefficients, we find:
$F(t) = 1 + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^2 + \ldots = \exp(tA)$. $\square$

Leibniz Rule for Power Series

Lemma

If $\phi(t)$ and $\psi(t)$ are power series in t with coefficients from $\Lambda^m V$ and $\Lambda^n V$ respectively, then the Leibniz rule holds, i.e.

$$\partial_t(\phi \wedge \psi) = (\partial_t \phi) \wedge \psi + \phi \wedge (\partial_t \phi)$$

Proof of Leibniz Rule for Power Series

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Proof.

Due to linearity of derivative and the fact that power series can be differentiated term by term, just check for $\phi = t^2\omega_1$ and $\psi = t^b\omega_2$.

$$\partial_t(\phi \wedge \psi) = (a+b)t^{a+b-1}\omega_1 \wedge \omega_2$$

$$(\partial_t \phi) \wedge \psi + \phi \wedge (\partial_t \psi) = at^{a-1}\omega_1 \wedge t^b\omega_2 + t^a\omega_1 \wedge bt^{b-1}\omega_2$$

□

Inverse

Lemma

The inverse of a formal power series $\phi(t)$ exists iff $\phi(0) \neq 0$.

Proof.

If $\phi(0) \neq 0$ then $\phi(t) = \phi(0) + t\psi(t)$ with ψ another power series. Then we can construct the inverse explicitly:

$$\frac{1}{\phi(t)} = \frac{1}{\phi(0)} \frac{1}{(1 + \frac{t\psi(t)}{\phi(0)})} = \sum_{n=0}^{\infty} (-1)^n \phi(0)^{-n-1} (t\psi(t))^n$$

This is because $1 = (1+x)(1 - x + x^2 - x^3 + \ldots)$ for formal x. □

Jacobi

Lemma (Jacobi's Formula)

If $A(t)$ is an invertible operator-valued formal power series:

$$\partial_t \det(A(t)) = \det(A) tr(A^{-1}\partial_t A)$$

Note that the determinant and trace of $A(t)$ still makes sense because $A(t)$ is still an operator (just expressed as an infinite sum of operators).

Note: this formula can be written in a lot of different ways (Cf. Wikipedia).

Jacobi

Lemma (Jacobi's Formula)

If A is an invertible operator-valued formal power series:

$$\partial_t \det(A(t)) = \det(A) tr(A^{-1}\partial_t A)$$

Proof.

Apply definition of determinant and the Leibniz rule established earlier.

$$(\partial_t \det(A(t)))(\omega) = \partial_t(\det(A(t))\omega) = \partial_t(Av_1 \wedge \dots \wedge Av_n)$$

$$= \sum_{k=1}^{n} Av_1 \wedge \dots \wedge (\partial_t A)v_k \wedge \dots \wedge Av_n$$

(We want to write this as a trace.) $\square$

Jacobi Proof, Continued

Proof, continued.

Invoke the algebraic complement of $A(t)$, given by $\det(A(t)) \cdot A^{-1}(t)$ for invertible A (there is a general formula as well). Think of it like the adjoint.

$$\sum_{k=1}^{n} Av_1 \wedge \ldots \wedge (\partial_t A)v_k \wedge \ldots \wedge Av_n = \sum_{k=1}^{n} v_1 \wedge \ldots \wedge (\tilde{A}\partial_t A v_k) \wedge \ldots \wedge v_n$$

Note that the right-hand side is a trace: $\Lambda^n(\tilde{A}\partial_t A)^1(v_1 \wedge \ldots \wedge v_n)$. This gives us the desired identity.

$$\partial_t \det(A) = \mathrm{tr}(\tilde{A}\partial_t A) = \mathrm{tr}(\det(A)A^{-1}\partial_t A) = \det(A)\mathrm{tr}(A^{-1}\partial_t A)$$

$\square$

Proof of Liouville

Let $F(t) = \exp(tA)$. $F(0) = 1$ so it is invertible, so we can apply Jacobi's:

$$\partial_t \det(F(t)) = \det(F(t)) \cdot \mathrm{tr}(F^{-1}\partial_t F)$$

By the characterization of $F(t) = \exp(tA)$, we have $F^{-1}(\partial_t F) = F^{-1}(FA) = (F^{-1}F)A = A$.

$$\partial_t \det(F(t)) = \det(F(t)) \cdot \mathrm{tr}(A)$$

$$\text{Let } f(t) = \det(F(t)) \qquad \partial_t f(t) = f(t) \cdot \mathrm{tr}(A)$$

By the characterization, $f(t) = \exp(t \cdot \mathrm{tr}(A))$. Hence:

$$\det(\exp(tA)) = \exp(t \cdot \mathrm{tr}(A)) \quad \square$$

Related Identities

Generalization of Liouville's: For $p \leq n = \dim(V)$ with $A \in \mathrm{End}(V)$. Liouville's is the special case where $p = n$

$$\Lambda^p(\exp(tA))^p = \exp(t(\Lambda^p A^1))$$

Sylvester's Theorem: For $A : V \mapsto W$, $B : W \mapsto V$, we have:

$$\det(I_V + BA) = \det(I_W + AB)$$

An intuitive taste of Jacobi

A nice vignette in Arnold's ODE textbook.
Observation: As $\epsilon \to 0$ we have:

$$\det(I + \epsilon A) = 1 + \epsilon \mathrm{tr}(A) + O(\epsilon^2)$$

We can view this easily via the eigenvalues of A, call them $\lambda_1, ..., \lambda_n$.

$$\det(I + \epsilon A) = \prod_{i=1}^{n} (1 + \epsilon \lambda_i)$$

Note that the zeroth order term is 1. The second term is (by Vieta) $\epsilon(\sum_{i=1}^{n} \lambda_i) = \epsilon \mathrm{tr}(A)$. Rest of the terms are order ϵ^2. $\square$

Conclusion

There are a number of ways to view the duality of trace and determinant.

- Definitions and constructions.
- Theorems and analytical connections.

Careful understanding of the basic constructions (wedge product, power series, differential equation for exp, etc) was key.

Works Cited

Linear Algebra via Exterior Products. Sergei Winitzki. Section 4.5.
Ordinary Differential Equations. VI Arnold. Section 16.